Exercices Module 1

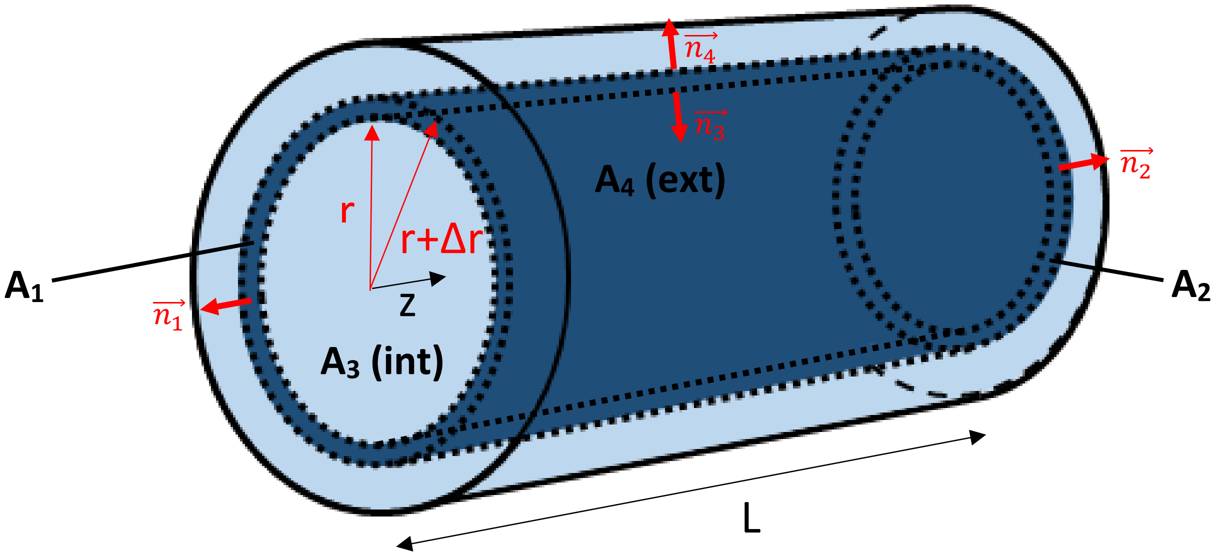
Correction 1.2 to 1.4

Notation used for module 1 exercise corrections:

Vectors are indicated as bold: $\boldsymbol{v}$

Basis vectors are indicated as $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$

Exercice 1.2



We neglect the effects of gravity on the system.

Momentum balance on the control volume (dark blue cylinder) reads:

$$\sum F_{volume} + \sum F_{surface} = \sum_{i=1}^{N} \rho \hat{\boldsymbol{v}} (\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{n}}) A_i$$

We have to consider pressure and shear forces on the surfaces of the control volume. In general, for pressure the force on surface i is $-\mathrm{p}_i \mathrm{A}_i\, \hat{\boldsymbol{\imath}}$ and the shear force on surface i due to fluid moving in the j direction is $-\tau_{ij}\big|_i (\widehat{\boldsymbol{n}_\imath} \cdot \hat{\boldsymbol{\imath}})\, \hat{\boldsymbol{\jmath}}\, \mathrm{A}_i$ note the force is in the j direction. Then considering pressure on surface 1 and 2 and shear on surfaces 3 and 4 we have:

$$-p_1 A_1\, \widehat{\boldsymbol{n_1}} - p_2 A_2\, \widehat{\boldsymbol{n_2}} + \tau_{rz}(r) A_3\, \hat{\boldsymbol{z}} - \tau_{rz}(r+\Delta r) A_4\, \hat{\boldsymbol{z}} = 0$$

<u>Note</u>: the sum of the forces is zero because the system is at steady state and the momentum of the fluid entering and exiting the control volume cancel.

The surface areas are:

$$A_1 = A_2 = \pi(r+\Delta r)^2 - \pi r^2 = 2\pi r \Delta r + \pi(\Delta r)^2 = 2\pi r \Delta r \text{ (becasue } \Delta \mathrm{r} \to 0)$$

$$A_3 = 2\pi L r$$

$$A_4 = 2\pi L(r + \Delta r)$$

Inserting the areas and also projecting onto the z-axis (direction of the flow):

$$p_1 2\pi r \Delta r - p_2 2\pi r \Delta r + \tau_{rz}(r) 2\pi L r - \tau_{rz}(r + \Delta r) 2\pi L(r + \Delta r) = 0$$

Dividing by $2\pi L \Delta r$ and replacing $p_1 - p_2 = \Delta p$

$$\frac{\Delta p}{L} r - \frac{(r\tau_{rz})|_{r+\Delta r} - (r\tau_{rz})|_r}{\Delta r} = 0$$

Therefore:

$$\boxed{\frac{d}{dr}(r\tau_{rz}) = \frac{\Delta p}{L} r}$$

Integrating with over r:

$$r\tau_{rz} = \frac{\Delta p}{2L} r^2 + C_1$$

$$\tau_{rz}(r) = \frac{\Delta p}{2L} r + \frac{C_1}{r}$$

When r = 0, physical quantities (including τ) cannot diverge, therefore C_1=0 and:

$$\boxed{\tau_{rz}(r) = \frac{\Delta p}{2L} r}$$

The relationship between the shear stress and the velocity profile is given by Newton's law:

$$\tau_{rz}(r) = -\mu \frac{dv_z}{dr}$$

So

$$\frac{dv_z}{dr} = -\frac{\Delta p}{2\mu L} r$$

Integrating again over r:

$$\boxed{v_z(r) = -\frac{\Delta p}{4\mu L} r^2 + C_2}$$

The boundary conditions give:

$$v_z(r = R) = 0$$

$$0 = -\frac{\Delta p}{4\mu L}R^2 + C_2$$

$$C_2 = \frac{\Delta p}{4\mu L}R^2$$

Therefore:

$$\boxed{\boldsymbol{v_z(r) = \frac{\Delta p}{4\mu L}(R^2 - r^2)}}$$

b) In the second case, the fluid is the same, and the forces applied on the control volume are also the same than in the first case. Only the boundary conditions are different, therefore the same differential equation is describing the flow:

$$\frac{d}{dr}(r\tau_{rz}) = \frac{\Delta p}{L}r$$

And so, the following is still true:

$$\tau_{rz}(r) = \frac{\Delta p}{2L}r + \frac{C_1}{r}$$

However, in this new situation, there is no fluid in r = 0, and so we cannot use the same boundary condition. Therefore, we have to keep C_1 in the expression, replace τ_{rz} using Newton's law of viscosity, and integrate over r:

$$-\mu\frac{dv_z}{dr} = \frac{\Delta p}{2L}r + \frac{C_1}{r}$$

$$\frac{dv_z}{dr} = -\frac{\Delta p}{2\mu L}r - \frac{C_1}{\mu r}$$

$$\boxed{v_z(r) = -\frac{\Delta p}{4\mu L}r^2 - \frac{C_1}{\mu}\ln(r) + C_2}$$

The new boundary conditions are:

$$\begin{cases} v_z(r = R_1) = 0 \\ v_z(r = R_2) = 0 \end{cases}$$

$$\begin{cases} -\frac{\Delta p}{4\mu L}R_1^2 - \frac{C_1}{\mu}\ln(R_1) + C_2 = 0 \ \ (\mathbf{1}) \\ -\frac{\Delta p}{4\mu L}R_2^2 - \frac{C_1}{\mu}\ln(R_2) + C_2 = 0 \ (\mathbf{2}) \end{cases}$$

(2) – (1) gives :

$$\frac{\Delta p}{4\mu L}(R_2^2 - R_1^2) + \frac{C_1}{\mu}\ln\left(\frac{R_2}{R_1}\right)$$

$$\boxed{C_1 = \frac{\Delta p}{4L}\frac{R_1^2 - R_2^2}{\ln\left(\frac{R_2}{R_1}\right)}}$$

Replacing in (1):

$$-\frac{\Delta p}{4\mu L}R_1^2 - \frac{\Delta p}{4\mu L}\frac{R_1^2 - R_2^2}{\ln\left(\frac{R_2}{R_1}\right)}\ln(R_1) + C_2 = 0$$

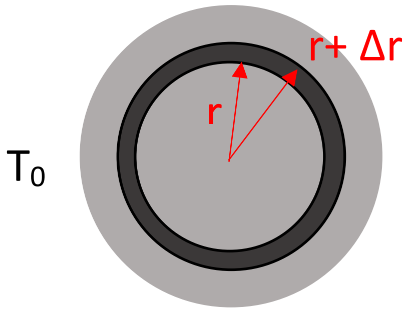
$$\boxed{C_2 = \frac{\Delta p}{4\mu L}\left[R_1^2 - \frac{R_2^2 - R_1^2}{\ln\left(\frac{R_2}{R_1}\right)}\ln(R_1)\right]}$$

Finally:

$$v_z(r) = -\frac{\Delta p}{4\mu L}r^2 - \frac{\Delta p}{4\mu L}\frac{R_1^2 - R_2^2}{\ln\left(\frac{R_2}{R_1}\right)}\ln(r) + \frac{\Delta p}{4\mu L}\left[R_1^2 - \frac{R_2^2 - R_1^2}{\ln\left(\frac{R_2}{R_1}\right)}\ln(R_1)\right]$$

$$\boxed{\boldsymbol{v_z(r) = \frac{\Delta p}{4\mu L}\left[R_1^2 - r^2 + (R_2^2 - R_1^2)\frac{\ln\left(\frac{r}{R_1}\right)}{\ln\left(\frac{R_2}{R_1}\right)}\right]}}$$

Exercice 1.3



In this situation, the temperature is maximum at the center of the sphere and $q(r)$ is radial from the center towards the surface.

Therefore, the heat balance on the shell control volume is:

$$Q_{in} - Q_{out} + Q_{created} = 0 \text{ (steady state)}$$

i.e.

$$q(r) * 4\pi r^2 - q(r + \Delta r) * 4\pi(r + \Delta r)^2 + S_n(r)V_{contrôle} = 0$$

The control volume is:

$$V_{control} = \frac{4}{3}\pi(r+\Delta r)^3 - \frac{4}{3}\pi r^3$$

$$V_{control} = \frac{4}{3}\pi(\cancel{r^3} + 3r^2\Delta r + 3r\Delta r^2 + \Delta r^3 - \cancel{r^3})$$

$$V_{control} = \frac{4}{3}\pi * 3r^2\Delta r = 4\pi r^2\Delta r \text{ (we can neglect the higher powers of } \Delta\text{r)}$$

So

$$4\pi[(r^2 q)|_r - (r^2 q)|_{r+\Delta r}] + S_{n0}\left(1 + b\left(\frac{r}{R}\right)^2\right)4\pi r^2\Delta r = 0$$

Dividing by $4\pi\Delta r$:

$$\boxed{\frac{d}{dr}(r^2 q) = S_{n0}\left(1 + b\left(\frac{r}{R}\right)^2\right)r^2}$$

Integrating over r:

$$r^2 q = S_{n0}\left(\frac{r^3}{3} + \frac{b}{5}\frac{r^5}{R^2}\right) + C_1$$

$$q(r) = S_{n0}\left(\frac{r}{3} + \frac{b}{5}\frac{r^3}{R^2}\right) + \frac{C_1}{r^2}$$

Since the flux cannot diverge for r=0, we have C_1=0.

Therefore:

$$\boxed{q(r) = S_{n0}\left(\frac{r}{3} + \frac{b}{5}\frac{r^3}{R^2}\right)}$$

According to Fourier's law:

$$q(r) = -k\frac{dT}{dr}$$

So

$$\frac{dT}{dr} = -\frac{S_{n0}}{k}\left(\frac{r}{3} + \frac{b}{5}\frac{r^3}{R^2}\right)$$

And

$$T(r) = -\frac{S_{n0}}{k}\left(\frac{r^2}{6} + \frac{b}{20}\frac{r^4}{R^2}\right) + C_2$$

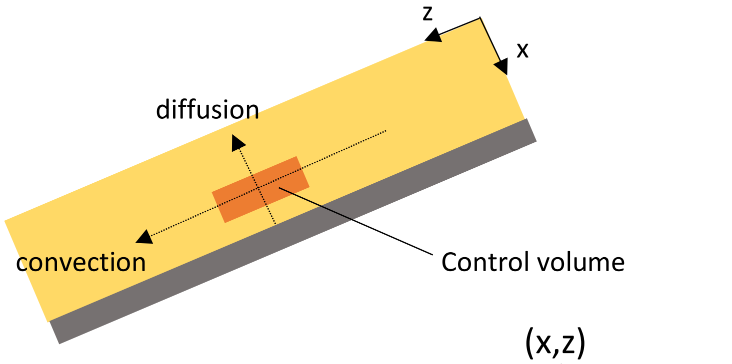
When r=R, T(R)=T_0, therefore:

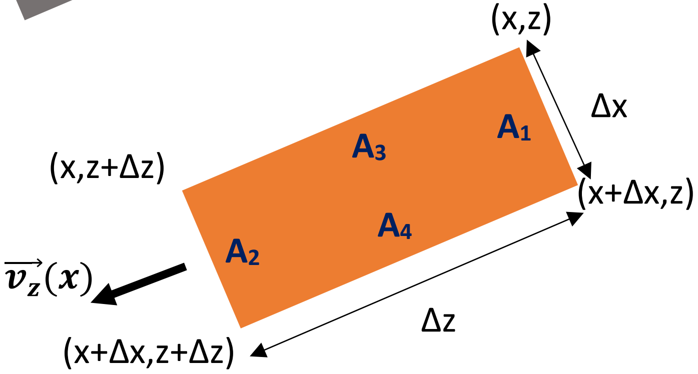
$$T_0 = -\frac{S_{n0}}{k}\left(\frac{R^2}{6} + \frac{b}{20}\frac{R^4}{R^2}\right) + C_2$$

$$\boxed{C_2 = \frac{S_{n0}R^2}{k}\left(\frac{1}{6} + \frac{b}{20}\right) + T_0}$$

Finally: $$\boxed{\boldsymbol{T(r) = T_0 + \frac{S_{n0}R^2}{k}\left[\frac{1}{6}\left(1 - \left(\frac{r}{R}\right)^2\right) + \frac{b}{20}\left(1 - \left(\frac{r}{R}\right)^4\right)\right]}}$$

Exercice 1.4





We consider that the transport of lead inside the oil happens by convection only along the z-axis (through A_1 and A_2) and by diffusion only along the x-axis (through A_3 and A_4). The molecular balance gives:

$$c_A(x,z)v_z(x)A_1 - c_A(x,z+\Delta z)v_z(x)A_2 - j_{A,x}(x+\Delta x,z)A_4 + j_{A,x}(x,z)A_3 = 0$$

Since in this case $j_{A,x} = -D_{AB}\frac{\partial c_A}{\partial x}$ (Fick's law) then we can simplify as:

$$c_A(x,z)v_z(x)W\Delta x - c_A(x,z+\Delta z)v_z(x)W\Delta x + D_{AB}\frac{\partial c_A}{\partial x}(x+\Delta x,z)W\Delta z - D_{AB}\frac{\partial c_A}{\partial x}(x,z)W\Delta z = 0$$

Dividing by $W\Delta x\Delta z$:

$$\frac{c_A(x,z) - c_A(x,z+\Delta z)}{\Delta z}v_z(x) + D_{AB}\frac{\frac{\partial c_A}{\partial x}(x+\Delta x,z) - \frac{\partial c_A}{\partial x}(x,z+\Delta z)}{\Delta x} = 0$$

Using partial derivatives:

$$-\frac{\partial c_A(x,z)}{\partial z} v_z(x) + D_{AB} \frac{\partial^2 c_A(x,z)}{\partial x^2} = 0$$

$$\boxed{\boldsymbol{D_{AB} \frac{\partial^2 c_A(x,z)}{\partial x^2} - \frac{\partial c_A(x,z)}{\partial z} * \frac{\rho g \delta^2 \cos\beta}{2\mu}\left[1 - \left(\frac{x}{\delta}\right)^2\right] = 0}}$$

Boundary conditions are:

$$c_A(x = \delta, z) = c_{A,sat}; \qquad \frac{\partial c_A}{\partial x}(x = 0, z) = 0; \qquad c_A(x, z = 0) = 0$$

Solving this using analytical methods is very difficult.